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Marginality and Triangle Inequality

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Abstract In this paper we study conditions for the existence of a 3-dimensional s-map on a quantum logic under assumption that marginal s-maps are known. We show that the existence of such a 3-dimensional s-map depends on the triangle inequality of d-map, which on a Boolean algebra represents a measure of symmetric difference.

Keywords Orthomodular lattice · State · s-map · Difference map · Triangle inequality

1 Introduction and preliminaries

There are various approaches to model random events. The classical measurable space with a normed measure (probability measure) is fundamental. In this case for each n-dimensional random vector there exists an n-dimensional joint distribution function. On the other hand it is not always easy to construct such a joint distribution function if marginal distribution functions are known, for more details see [3, 13].

Additionally it is a well known fact, that there are cases in which it is not possible to construct a 3-dimensional distribution function from given 2-dimensional marginal distribution functions. Hence it follows that such random variables do not belong to the same probability space. Such a situation can be modelled in different ways. We use an orthomodular lattice $\mathcal{L}$ with a state m (a quantum logic) [5, 6, 14, 16]. Typical examples are: a horizontal sum of Boolean algebras (σ-algebras) and projection lattices of von Neumann algebras [15].

In this paper, we use a function of n-variables (s-map) as a model of a joint distribution function ([8, 9]). s-map as a function of 2-variables was introduced in [7] and it has been studied, for example, in [1, 2, 4, 8–10]. s-map as a multivariable function was introduced in [9]. We study under which conditions is the s-map marginal. This problem is related to the

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fact, that for three random variables with known distribution functions, a joint distribution function may not exist. The solution for this problem depends on the triangle inequality of another map, called d-map, which on a Boolean algebra represents a measure of a symmetric difference.

In this paper we assume an orthomodular lattice $\mathcal{L}$ with a state m as a basic model. This structure is also called a quantum logic. Let us recall the existence of orthomodular lattices without any state [5, 12].

In this chapter we briefly recapitulate basic terms such as orthomodular lattice, orthogonal and compatible elements and a state. For more information see [6, 14, 16].

Definition 1 Let L be a lattice (a nonempty set endowed with a partial ordering $\leq$, the lattice operations supremum $\vee$ and infimum $\wedge$) with the greatest element I and the smallest element O. Let $\perp: L \to L$ be a unary operation on L with the following properties:

1. For all $a \in L$ there exists a unique $a^\perp \in L$ such that $(a^\perp)^\perp = a$ and $a \vee a^\perp = I$
2. If $a, b \in L$ and $a \leq b$ then $b^\perp \leq a^\perp$
3. If $a, b \in L$ and $a \leq b$ then $b = a \vee (a^\perp \wedge b)$ (orthomodular law).

Then $\mathcal{L} = (L, O, I, \vee, \wedge, \perp)$ is an orthomodular lattice.

Definition 2 Let $\mathcal{L}$ be an orthomodular lattice. Then the elements $a, b \in L$ are called:

1. orthogonal ($a \perp b$) if $a \leq b^\perp$;
2. compatible ($a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b^\perp)$ and $b = (a \wedge b) \vee (a^\perp \wedge b)$.

Definition 3 Let $\mathcal{L}$ be an orthomodular lattice. A map $m : L \to [0, 1]$ satisfying the following conditions:

1. $m(O) = 0$ and $m(I) = 1$
2. If $a \perp b$; $a, b \in L$ then $m(a \vee b) = m(a) + m(b)$

is called a state on L.

Definition 4 A quantum logic is an orthomodular lattice with at least one state.

Definition 5 Let $\mathcal{L}$ be a quantum logic. An s-map on L is a map $p : L^n \to [0, 1], n \in N$ satisfying the following conditions:

(s1) $p(I, \dots, I) = 1$;
(s2) if $a_i \perp a_{i+1}$ for some $i \in \{1, 2, \dots, n-1\}$, then $p(a_1, \dots, a_n) = 0$;
(s3) if $a_i \perp b_i$ for some $i \in \{1, 2, \dots, n\}$, then

$$p(a_1, \dots, a_i \vee b_i, \dots, a_n) = p(a_1, \dots, a_i, \dots, a_n) + p(a_1, \dots, b_i, \dots, a_n).$$

Proposition 1 [9] *Let $\mathcal{L}$ be a quantum logic and let p be an s-map on L. Then*

(1) *if $a_i \perp a_j$ for some $i, j \in \{1, 2, \dots, n\}$, then $p(a_1, \dots, a_n) = 0$;*
(2) *a map $\nu : L \to [0, 1]$, $\nu(a) := p(a, \dots, a)$ is a state on L;*
(3) *for any $(a_1, \dots, a_n) \in L^n : p(a_1, \dots, a_n) \leq \nu(a_i)$ for each $i \in \{1, 2, \dots, n\}$;*
(4) *if $a_i \leftrightarrow a_j$ for some $i, j \in \{1, 2, \dots, n\}$, then*

$$p(a_1, \dots, a_n) = p(a_1, \dots, a_{i-1}, a_i \wedge a_j, a_{i+1}, \dots, a_{j-1}, a_i \wedge a_j, a_{j+1}, \dots, a_n).$$

Let us denote $\pi(a_1, \ldots, a_n)$ a permutation of $(a_1, \ldots, a_n)$. In general an s-map on L is not invariant with respect to permutations, i.e.

$$p(a_1, a_2, \ldots, a_n) = p(\pi(a_1, a_2, \ldots, a_n))$$

need not be fulfilled, see [9].

Proposition 2 [9] *Let $\mathcal{L}$ be a quantum logic, let p be an s-map on L and let $(a_1, \ldots, a_n) \in L^n$.*

(1) *If $a_i = I$ for some $i \in \{1, 2, \ldots, n\}$, then*

$$p(a_1, \ldots, a_n) = p(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_n)$$

for each $j \in \{1, 2, \ldots, n\}$;

(2) *If $a_i \leftrightarrow a_j$ for some $i, j \in \{1, 2, \ldots, n\}$, then*

$$p(a_1, \ldots, a_n) = p(\pi(a_1, \ldots, a_n)).$$

Let $\mathcal{L}$ be a quantum logic and let p be an n-dimensional s-map on L. Then it is easy to prove, that for $k \leq n$ the function $p_k : L^k \to [0, 1]$, $p_k(a_1, \ldots, a_k) = p(a_1, \ldots, a_k, I, \ldots, I)$ is a k-dimensional s-map on L. Such an s-map is called a marginal s-map. The next lemma follows directly from the previous proposition and from the fact, that each element of L is compatible with I.

Lemma 1 *Each marginal s-map is invariant with respect to permutations.*

2 Extension of s-map

As we mentioned in the Introduction, our goal is to determine conditions for the extension of the dimension of an s-map. In this chapter we study whether a 2-dimensional s-map can be extended to a 3-dimensional one. To solve this problem we use another map called a difference map [2, 11].

Definition 6 Let $\mathcal{L}$ be a quantum logic. A difference map on L (d-map) is a map $d : L^2 \to [0, 1]$ satisfying the following conditions:

(d1) $d(I, I) = 0$ and $d(O, I) = d(I, O) = 1$;

(d2) if $a \perp b$ then $d(a, b) = d(a, O) + d(O, b)$;

(d3) if $a \perp b$ then for each $c \in L$

$$d(a \vee b, c) = d(a, c) + d(b, c) - d(O, c)$$

$$d(c, a \vee b) = d(c, a) + d(c, b) - d(c, O).$$

Simple assertions in the next lemma will be helpful in several proofs.

Lemma 2 *Let $\mathcal{L}$ be a quantum logic and let p be an s-map on L. Then for any $a, b \in L$:*

(1) $p(a, I) = p(I, a) = p(a, a)$,
(2) $p(a^{\perp}, a^{\perp}) = 1 - p(a, a)$,
(3) $p(a^{\perp}, b) = p(b, b) - p(a, b)$ *and* $p(a, b^{\perp}) = p(a, a) - p(a, b)$.

Proof

(1) $p(a, I) = p(a, a \vee a^{\perp}) = p(a, a) + p(a, a^{\perp}) = p(a, a)$,
(2) $1 = p(I, I) = p(a \vee a^{\perp}, I) = p(a, I) + p(a^{\perp}, I) = p(a, a) + p(a^{\perp}, a^{\perp})$,
(3) $p(b, b) = p(I, b) = p(a \vee a^{\perp}, b) = p(a, b) + p(a^{\perp}, b)$. □

Lemma 3 *Let $\mathcal{L}$ be a quantum logic and let $p : L^2 \to [0, 1]$ be an s-map. Then the map $d_p : L^2 \to [0, 1]$, $d_p(a, b) = p(a, b^{\perp}) + p(a^{\perp}, b)$ is a d-map.*

Proof d_p satisfies all requirements of the Definition 6
1. $d_p(I, I) = p(I, O) + p(O, I) = 0, d_p(O, I) = p(O, O) + p(I, I) = 1$
2. if $a \perp b$ then

$$\begin{aligned} d_p(a, O) + d_p(O, b) &= p(a, I) + p(a^{\perp}, O) + p(O, b^{\perp}) + p(I, b) \\ &= p(a, b) + p(a, b^{\perp}) + p(a, b) + p(a^{\perp}, b) \\ &= d_p(a, b) \end{aligned}$$

3. if $a \perp b$ then

$$\begin{aligned} d_p(a \vee b, c) &= p(a \vee b, c^{\perp}) + p((a \vee b)^{\perp}, c) \\ &= p(a \vee b, c^{\perp}) + (p(c, c) - p(a \vee b, c)) + p(c, c) - p(c, c) \\ &= p(a, c^{\perp}) + p(b, c^{\perp}) + p(c, c) - p(a, c) + p(c, c) - p(b, c) - p(c, c) \\ &= p(a, c^{\perp}) + p(b, c^{\perp}) + p(a^{\perp}, c) + p(b^{\perp}, c) - (p(O, c^{\perp}) + p(I, c)) \\ &= d_p(a, c) + d_p(b, c) - d_p(O, c). \end{aligned}$$

The notation d_p will be used when the d-map is induced by the s-map p. □

Lemma 4 *Let $\mathcal{L}$ be a quantum logic, let p be an s-map on L and d_p be the d-map induced by the s-map p. Then d_p satisfies the triangle inequality if and only if p has the following property*:

$$0 \leq p(a, b) + p(c, c) - p(a, c) - p(c, b) \quad \forall a, b, c \in L.$$

Proof Let d_p be the d-map induced by the s-map p. Then d_p satisfies the triangle inequality if and only if

$$\forall a, b, c \in L : d_p(a, b) \leq d_p(a, c) + d_p(c, b),$$

which is equivalent to

$$p(a, b^{\perp}) + p(a^{\perp}, b) \leq p(a, c^{\perp}) + p(a^{\perp}, c) + p(c, b^{\perp}) + p(c^{\perp}, b).$$

Now we apply Lemma 2 and we get

$$\begin{aligned} (p(a, a) - p(a, b)) + (p(b, b) - p(a, b)) \leq{}& (p(a, a) - p(a, c) + (p(c, c) - p(a, c)) \\ &+ (p(c, c) - p(c, b)) + (p(b, b) - p(c, b)), \end{aligned}$$

and then

$$0 \le p(a,b) + p(c,c) - p(a,c) - p(c,b).$$ □

We denote the expression $p(a,b) + p(c,c) - p(a,c) - p(c,b)$ by $T(a,b,c)$. Then the triangle inequality has the form

$$0 \le T(a,b,c).$$

Clearly, if p is invariant w.r.t. permutations, then $T(a,b,c) = T(b,a,c)$ for any $a,b,c \in L$.

Lemma 5 *Let $\mathcal{L}$ be a quantum logic and let p be an s-map on L which is invariant w.r.t. permutations. Then for any $a,b,c \in L$:*

$$T(a,b,c) = T(a^\perp,b^\perp,c^\perp) = T(a^\perp,c,b) = T(a,c^\perp,b^\perp) = T(c,b^\perp,a) = T(c^\perp,b,a^\perp).$$

Proof At first we prove: $T(a^\perp,c,b) = T(a,b,c)$. We use Lemma 2 and the invariance of p.

$$\begin{aligned} T(a^\perp,c,b) &= p(a^\perp,c) + p(b,b) - p(a^\perp,b) - p(b,c) \\ &= (p(c,c) - p(a,c)) + p(b,b) - (p(b,b) - p(a,b)) - p(b,c) \\ &= p(a,b) + p(c,c) - p(a,c) - p(b,c) \\ &= p(a,b) + p(c,c) - p(a,c) - p(c,b) \\ &= T(a,b,c). \end{aligned}$$

Now we alternately apply the former property and the invariance of T w.r.t. the order of the first and second coordinates to prove the remaining equalities.

$$\begin{aligned} T(a,b,c) &= T(a^\perp,c,b) = T(c,a^\perp,b) = T(c^\perp,b,a^\perp) \\ &= T(b,c^\perp,a^\perp) = T(b^\perp,a^\perp,c^\perp) = T(a^\perp,b^\perp,c^\perp). \end{aligned}$$

And finally, $T(a,b,c) = T(a^\perp,b^\perp,c^\perp)$ implies $T(a^\perp,c,b) = T(a,c^\perp,b^\perp)$ and $T(c^\perp,b,a^\perp) = T(c,b^\perp,a)$. □

Lemma 6 *Let $\mathcal{L}$ be a quantum logic, in which $L = \{a,a^\perp,b,b^\perp,c,c^\perp,O,I\}$, let p be an s-map on L which is invariant w.r.t. permutations and let d_p be the d-map induced by p. Then d_p satisfies the triangle inequality if and only if the following four inequalities are fulfilled:*

$$0 \le T(a,b,c),$$

$$0 \le T(b,c,a),$$

$$0 \le T(c,a,b),$$

$$T(a,b,c) + T(b,c,a) + T(c,a,b) \le 1.$$

Proof The triangle inequality $0 \leq T(x, y, z)$ is always satisfied on L if the triple (x, y, z) contains

(a) two identical elements or
(b) the minimal or the maximal element or
(c) an element and its orthocomplement.

The key role play the triples consisting of distinguished elements of L, which belong to none of the previous classes. There are $2.2.2.3! = 48$ of such triples (x, y, z). Since p is invariant w.r.t. permutations, $T(x, y, z) = T(y, x, z)$ for any $x, y, z \in L$. Therefore we consider 24 triples, which we partition to four classes:

$$\{(a, b, c), (a^\perp, b^\perp, c^\perp), (a^\perp, c, b), (a, c^\perp, b^\perp), (c, b^\perp, a), (c^\perp, b, a^\perp)\},$$
$$\{(b, c, a), (b^\perp, c^\perp, a^\perp), (b^\perp, a, c), (b, a^\perp, c^\perp), (a, c^\perp, b), (a^\perp, c, b^\perp)\},$$
$$\{(c, a, b), (c^\perp, a^\perp, b^\perp), (c^\perp, b, a), (c, b^\perp, a^\perp), (b, a^\perp, c), (b^\perp, a, c^\perp)\},$$
$$\{(a, b, c^\perp), (a^\perp, b^\perp, c), (a^\perp, c^\perp, b), (a, c, b^\perp), (c^\perp, b^\perp, a), (c, b, a^\perp)\}.$$

We will show that all triples in one class represent the only one triangle inequality from our lemma. According to Lemma 5

$$T(a, b, c) = T(a^\perp, b^\perp, c^\perp) = T(a^\perp, c, b) = T(a, c^\perp, b^\perp) = T(c, b^\perp, a) = T(c^\perp, b, a^\perp),$$

therefore this 6 inequalities

$$0 \leq T(a, b, c), 0 \leq T(a^\perp, b^\perp, c^\perp), 0 \leq T(a^\perp, c, b),$$
$$0 \leq T(a, c^\perp, b^\perp), 0 \leq T(c, b^\perp, a), 0 \leq T(c^\perp, b, a^\perp)$$

are equivalent. The first class of triples corresponds to the first inequality in our lemma. Situation is analogous for the second and also the third inequality. The last inequality $T(a, b, c) + T(b, c, a) + T(c, a, b) \leq 1$ is equivalent to $0 \leq T(a, b, c^\perp)$, because $1 - (T(a, b, c) + T(b, c, a) + T(c, a, b)) = T(a, b, c^\perp)$, what we will demonstrate shortly. We apply Lemma 2 and the invariance of p.

$$\begin{aligned}1 - (T(a, b, c) + T(b, c, a) + T(c, a, b)) &= 1 - (p(a, b) + p(c, c) - p(a, c) - p(c, b) \\ &\quad + p(b, c) + p(a, a) - p(b, a) - p(a, c) \\ &\quad + p(c, a) + p(b, b) - p(c, b) - p(b, a)) \\ &= 1 + p(a, b) + p(a, c) + p(b, c) \\ &\quad - p(a, a) - p(b, b) - p(c, c).\end{aligned}$$

$$\begin{aligned}T(a, b, c^\perp) &= p(a, b) + p(c^\perp, c^\perp) - p(a, c^\perp) - p(c^\perp, b) \\ &= p(a, b) + 1 - p(c, c) - (p(a, a) - p(a, c)) - (p(b, b) - p(c, b)) \\ &= 1 + p(a, b) + p(a, c) + p(b, c) - p(a, a) - p(b, b) - p(c, c).\end{aligned}$$

Therefore the last class of triples corresponds to the last inequality in our lemma. □

Corollary 1 *Let $\mathcal{L}$ be a horizontal sum of Boolean algebras $\{a_i, a_i^{\perp}, O, I\}, i \in \{1, 2, \ldots, n\}$, let p be an s-map on L which is invariant w.r.t. permutations and let d_p be the d-map induced by p. Then d_p satisfies the triangle inequality if and only if for all $a_i, a_j, a_k \in \{a_1, a_2, \ldots a_n\}, i \neq j, j \neq k, k \neq i$:*

$$0 \leq T(a_i, a_j, a_k),$$

$$T(a_i, a_j, a_k) + T(a_j, a_k, a_i) + T(a_k, a_i, a_j) \leq 1.$$

Lemma 7 *Let $\mathcal{L}$ be a quantum logic. Each d-map d_p, which is induced by a marginal 2-dimensional s-map p on L , satisfies the triangle inequality.*

Proof Let p_2 be marginal 2-dimensional s-map on L. This means, that there exists an s-map $p_3 : L^3 \to [0, 1]$ such that $p_2(a, b) = p_3(a, b, I)$ $\forall a, b \in L$. Let d_{p_2} be the d-map induced by the s-map p_2. Then for each $a, b \in L$

$$d_{p_2}(a, b) = p_2(a, b^{\perp}) + p_2(a^{\perp}, b) = p_3(a, b^{\perp}, I) + p_3(a^{\perp}, b, I).$$

From Proposition 1 and Proposition 2 we get for any $c \in L$

$$d_{p_2}(a, b) = p_3(a, b^{\perp}, c) + p_3(a, b^{\perp}, c^{\perp}) + p_3(a^{\perp}, b, c) + p_3(a^{\perp}, b, c^{\perp}).$$

Moreover

$$p_3(a, b^{\perp}, c) \leq p_2(b^{\perp}, c)$$
$$p_3(a^{\perp}, b, c^{\perp}) \leq p_2(b, c^{\perp})$$
$$p_3(a, b^{\perp}, c^{\perp}) \leq p_2(a, c^{\perp})$$
$$p_3(a^{\perp}, b, c) \leq p_2(a^{\perp}, c).$$

Hence

$$d_{p_2}(a, b) \leq p_2(a, c^{\perp}) + p_2(a^{\perp}, c) + p_2(b^{\perp}, c) + p_2(b, c^{\perp}) = d_{p_2}(a, c) + d_{p_2}(c, b). \quad \square$$

Comment 1 It is proved in [10] that a von Neumann algebra admits a unique 2-dimensional s-map

$$p(A, B) = \frac{tr(AB)}{tr(I)},$$

where I is the identity operator. It is not difficult to show that the triangle inequality is not valid for the d-map d_p which is induced by this s-map p. By the previous lemma, we may deduce that a von Neumann algebra admits no 3-dimensional s-map.

Proposition 3 *Let $\mathcal{L}$ be a quantum logic, in which $L = \{a, a^{\perp}, b, b^{\perp}, c, c^{\perp}, O, I\}$, let p_2 be a 2-dimensional s-map on L which is invariant w.r.t. permutations and let d_{p_2} be the d-map induced by p_2. The s-map p_2 is a marginal one if and only if d_{p_2} satisfies the triangle inequality on L.*

Proof The fact, that the triangle inequality is the necessary condition for an s-map on L to be marginal results from Lemma 7. We will prove that it is also the sufficient condition.

Let p_2 be an s-map on L which is invariant w.r.t. permutations and let d_{p_2} satisfies the triangle inequality on L. Let $T(x,y,z) = p_2(x,y) + p_2(z,z) - p_2(x,z) - p_2(z,y)$. We define a map $p_3 : L^3 \to [0,1]$ in the following way:

$$p_3(x,y,z) = 0 \quad \textit{if } x \perp y \quad \textit{or} \quad y \perp z \quad \textit{or} \quad z \perp x,$$

$$p_3(\pi(x,y,e)) = p_2(x,y) \quad \textit{if } e \in \{I, x, y\},$$

and

$$\begin{aligned}
p_3(\pi(a,b,c)) &= \alpha \\
p_3(\pi(a^\perp,b,c)) &= p_2(b,c) - \alpha \\
p_3(\pi(a,b^\perp,c)) &= p_2(a,c) - \alpha \\
p_3(\pi(a,b,c^\perp)) &= p_2(a,b) - \alpha \\
p_3(\pi(a^\perp,b^\perp,c)) &= T(a,b,c) - p_2(a,b) + \alpha \\
p_3(\pi(a^\perp,b,c^\perp)) &= T(c,a,b) - p_2(a,c) + \alpha \\
p_3(\pi(a,b^\perp,c^\perp)) &= T(b,c,a) - p_2(b,c) + \alpha \\
p_3(\pi(a^\perp,b^\perp,c^\perp)) &= 1 - T(a,b,c) - T(b,c,a) - T(c,a,b) - \alpha.
\end{aligned}$$

We prove that p_3 is an s-map extending p_2. The following statements result directly from the definition of p_3:

(a) p_3 has the properties (s1)–(s3) of Definition 5,
(b) p_3 is invariant w.r.t. permutations,
(c) p_3 is an extension of p_2.

Therefore the only condition for p_3 to be an s-map on L is: $p_3(x,y,z) \in [0,1]$ for all $(x,y,z) \in L^3$. Now, we show that $p_3(x,y,z) \in [0,1]$ for all $(x,y,z) \in L^3$. It is sufficient to deal with the restricted set of triples

$$A = \{(a,b,c), (a^\perp,b,c), (a,b^\perp,c), (a,b,c^\perp), (a^\perp,b^\perp,c), (a^\perp,b,c^\perp), (a,b^\perp,c^\perp), (a^\perp,b^\perp,c^\perp)\},$$

because the values of $p_3(x,y,z)$ for the other triples are clearly in $[0,1]$.

So, we have the system of 8 inequalities of the form $0 \le p_3(x,y,z) \le 1$, where $(x,y,z) \in A$, which is equivalent to the system of linear inequalities for α, expressed in the following way:

$$\text{Max} \quad LS \le \alpha \le \text{Min} \quad RS,$$

where LS denotes the set of all left sides of inequalities

$$\begin{aligned}
LS = \{&0, p_2(a,b) - 1, p_2(a,c) - 1, p_2(b,c) - 1, p_2(a,b) - T(a,b,c), \\
&p_2(a,c) - T(c,a,b), p_2(b,c) - T(b,c,a), -(T(a,b,c) + T(b,c,a) + T(c,a,b))\}
\end{aligned}$$

and RS the set of all right sides

$$RS = \{1, p_2(a,b), p_2(a,c), p_2(b,c), 1 + p_2(a,b) - T(a,b,c), 1 + p_2(a,c) - T(c,a,b), \\ 1 + p_2(b,c) - T(b,c,a), 1 - (T(a,b,c) + T(b,c,a) + T(c,a,b))\}.$$

After excluding the elements, which evidently can not be maximum of LS, resp. minimum of RS we get

$$\text{Max}\left\{\begin{matrix} 0 \\ p_2(a,b) - T(a,b,c) \\ p_2(a,c) - T(c,a,b) \\ p_2(b,c) - T(b,c,a) \end{matrix}\right\} \le \alpha \le \text{Min}\left\{\begin{matrix} p_2(a,b) \\ p_2(a,c) \\ p_2(b,c) \\ 1 - (T(a,b,c) + T(b,c,a) + T(c,a,b)) \end{matrix}\right\}$$

Appropriate α exists, if each value from the left side is less than or equal to each value from the right side. That is, we have 16 inequalities, 12 of them are always fulfilled, because p_2 is an s-map on L. The remaining four inequalities

$$0 \le 1 - (T(a,b,c) + T(b,c,a) + T(c,a,b))$$

$$p_2(a,b) - T(a,b,c) \le p_2(a,b)$$

$$p_2(a,c) - T(c,a,b) \le p_2(a,c)$$

$$p_2(b,c) - T(b,c,a) \le p_2(b,c)$$

are according to Lemma 6 equivalent to the triangle inequality of d_{p_2} on L, so they are fulfilled too. Therefore p_3 is an s-map on L with p_2 as its marginal s-map. □

Comment 2 Notice that the map p_3 constructed in the former proof is invariant w.r.t. permutations. The way to construct a non-invariant p_3 is to take distinguished α_i as values of distinguished permutations of a, b, c.

Corollary 2 *Let $\mathcal{L}$ be a horizontal sum of Boolean algebras $\{a_i, a_i^\perp, O, I\}$, $i \in \{1, 2, \ldots, n\}$ and let p_2 be 2-dimensional s-map on L , which is invariant w.r.t. permutations. An s-map p_2 is a marginal one if and only if d_{p_2} satisfies the triangle inequality on L.*

Proof We do not perform the entire proof, because it is analogical to the previous one. The map p_3, which is crucial in the proof, can be defined in the following way: $p_3 : L^3 \to [0,1]$,

$$p_3(x,y,z) = 0 \quad \textit{if } x \perp y \quad \textit{or} \quad y \perp z \quad \textit{or} \quad z \perp x,$$

$$p_3(\pi(x,y,e)) = p_2(x,y) \quad \textit{if } e \in \{I, x, y\},$$

and if $a_i, a_j, a_k \in \{a_1, a_2, \ldots, a_n\}$, $i \neq j$, $j \neq k$, $k \neq i$ then

$$p_3(\pi(a_i, a_j, a_k)) = \alpha$$

$$p_3(\pi(a_i^\perp, a_j, a_k)) = p_2(a_j, a_k) - \alpha$$

$$p_3(\pi(a_i^\perp, a_j^\perp, a_k)) = T(a_i, a_j, a_k) - p_2(a_i, a_j) + \alpha$$

$$p_3(\pi(a_i^\perp, a_j^\perp, a_k^\perp)) = 1 - T(a_i, a_j, a_k) - T(a_j, a_k, a_i) - T(a_k, a_i, a_j) - \alpha.$$

Applying the analogical proceeding as in the previous proof we get: for $a_i, a_j, a_k \in \{a_1, a_2, \dots, a_n\}, i \neq j, j \neq k, k \neq i$:

$$0 \le T(a_i, a_j, a_k),$$

$$T(a_i, a_j, a_k) + T(a_j, a_k, a_i) + T(a_k, a_i, a_j) \le 1,$$

which are according to Corollary 1 equivalent to the triangle inequality of d_{p_2} on L. □

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